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Energy–Entropy Inequalities for Classical Lattice Systems

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A new characterization of equilibrium states for classical lattice systems is given in terms of correlation inequalities. Their physical meaning is found to express thermodynamic stability. We demonstrate the applicability of the inequalities in specific models.

KEY WORDS: Equilibrium State; Correlation Inequalities; Symmetry Breaking; Ising Systems.

1. INTRODUCTION

Recently new rigorous results on absence of long-range order, absence of symmetry breaking, etc. in statistical mechanics have been obtained (see Refs. 1–8).

These results often rely on the use of the Bogoliubov inequality when it is available (i.e., for quantum systems or for classical systems with a Poisson bracket structure) or on considerations dealing with the energy of configurations (energy versus entropy balance).

This paper emerged from a concern of how these considerations about thermodynamic stability might be understood in a more systematic way. For classical lattice systems we prove that the stability can be expressed by means of correlation inequalities, which we derive in Section 2 from the DLR[12] equations. The setup and the derivations are performed immediately for the infinite system. Moreover we prove that these correlation inequalities form in a way a complete set in the sense that if any state

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satisfies these inequalities, the state must necessarily also satisfy the DLR equations. It means that our correlation inequalities characterize fully the equilibrium states.

In Section 3 we give an alternative derivation from the variational principle. Of course, this method works only for translation invariant states, but it enhances the physical meaning of these inequalities as consequences of stability of the free energy against dissipative perturbations of a generalized Glauber type. This leads to the interpretation of the inequalities as expressing a balance between the entropy change against the energy change. One might also remark on the formal analogy of our inequalities with the quantum mechanical ones.[9-11]

Finally in a last section we demonstrate the technical power of the inequalities in specific models. Not only do they reproduce easily the sharpest results but also they give a more systematic understanding of *a priori* different techniques or phenomena.

Finally let us define our scheme. We consider the lattice $\mathbb{Z}^\nu$. At each site $j \in \mathbb{Z}^\nu$ we associate a copy K_j of some configuration space K, where K is assumed to be a compact Hausdorff space.

Given $X \subseteq \mathbb{Z}^\nu$, let K^X be the product $\prod\ (j \in X)\ K_j$ and denote $K^\infty = K^{\mathbb{Z}^\nu}$. To each bounded subset $\Lambda \subset \mathbb{Z}^\nu$ there corresponds a finite system in Λ with configuration space K^Λ, algebra of observables $C(K^\Lambda)$ (= the set of real-valued continuous functions on K^Λ). Consider a normalized regular Borel measure ρ_0 on K and denote also by $\rho_0(dx)$, $x \in K^\Lambda$ the product measure on $C(K^\Lambda)$.

The interaction is given by a set of functions $\phi(X)$, $X \subset \mathbb{Z}^\nu$, such that $\phi(X) \in C(K^X)$ and which are translation invariant, i.e., if $\tau_i\ (i \in \mathbb{Z}^\nu)$ represents the translation automorphism then $\phi(X+i) = \tau_i \phi(X)$. Finally we impose the condition that

$$\begin{aligned} |||\phi||| &= \sum_{0 \in X} \|\phi(X)\| < \infty \\ \|\phi(X)\| &= \sup_{x \in K^X} |\phi(X)(x)| \end{aligned} \tag{1}$$

The local Hamiltonians of the system are then defined as

$$H_\Lambda = \sum_{X \subset \Lambda} \phi(X)$$

2. DLR EQUATION AND CORRELATION INEQUALITIES

A way of defining equilibrium states (not necessarily translation invariant) is by means of the DLR equations.[12] These equations are conditions on the probability distributions of the equilibrium state, relative to fixed boundary conditions. Those probability measures are supposed to be absolutely continuous with respect to the *a priori* measure ρ_0.

Definition 2.1. Let ρ be a state of $C(K^\infty)$, then ρ satisfies the DLR equations for the interaction ϕ, if (i) for all finite $\Lambda \subset \mathbb{Z}^\nu$ the measure $\rho(s, d\tau)$ on K^{Λ^c} is uniquely defined for all $s \in K^\Lambda$ such that

$$\rho(f) = \int_{K^\Lambda}\int_{K^{\Lambda^c}} f(s \times \tau)\rho(s, d\tau)\rho_0(ds) \qquad \text{for all} \quad f \in C(K^\infty)$$

(ii) for all $\Lambda \subset \mathbb{Z}^\nu$ and $s_1, s_2 \in K^\Lambda$ and boundary condition $\tau \in K^{\Lambda^c}$ the measure $\rho(s, d\tau)$ satisfies

$$\rho(s_1, d\tau) = \exp\left[-{}_\tau H_\Lambda(s_1) + {}_\tau H_\Lambda(s_2)\right]\rho(s_2, d_\tau)$$

where $\tau H_\Lambda(s) = \sum_{X \cap \Lambda \neq \phi}\phi(X)(s \times \tau)$.

It is well known that for translation-invariant states the definition of equilibrium states by means of the DLR equations is equivalent with the one given by means of tangent functionals to the pressure or in terms of the variational principle (see next section).

In Sections 2 and 3 we absorb the inverse temperature in the interaction. Denote by Q the set of ρ_0-invariant, invertible maps U of K^∞ onto K^∞ such that $(Ux)_i = x_i$ for all i outside some finite $\Lambda \subset \mathbb{Z}^\nu$. For any $U \in Q$ define $\tilde{U}$ on $C(K^\infty)$ by

$$(\tilde{U}f)(x) = f(U^{-1}x), \qquad f \in C(K^\infty), \quad x \in K^\infty$$

and denote $\tilde{U}H - H = \lim_{\Lambda'}(\tilde{U}H_{\Lambda'} - H_{\Lambda'})$

Theorem 2.2. If ρ satisfies the DLR equations (2.1) then for all $U \in Q$ and $f \in C(K^\infty), f \geqslant 0, f \neq 0$ we have

$$\rho(f)\ln\frac{\rho(f)}{\rho(\tilde{U}f)} \leqslant \rho\big(f(\tilde{U}^{-1}H - H)\big)$$

Proof. Take any positive f in $C(K^\infty)$ and $U \in Q$. Then there exists a finite $\Lambda \subset \mathbb{Z}^\nu$ such that K^{Λ^c} is pointwise invariant under U. It is easily checked from 2.1 that $\rho(f) > 0$ and

$$\begin{aligned}
\rho(\tilde{U}f) &= \int_{K^\Lambda}\int_{K^{\Lambda^c}} f\big(U^{-1}(s \times \tau)\big)\rho(s, d\tau)\rho_0(ds)\\
&= \int_{K^\Lambda}\int_{K^{\Lambda^c}} f(U^{-1}s \times \tau)\rho(s, d\tau)\rho_0(ds)\\
&= \int_{K^\Lambda}\int_{K^{\Lambda^c}} f(s \times \tau)\rho(Us, d\tau)\rho_0(ds)\\
&= \int_{K^\Lambda}\int_{K^{\Lambda^c}} f(s \times \tau)\rho(s, d\tau)\exp\left[{}_\tau H_\Lambda(s) - {}_\tau H_\Lambda(Us)\right]\rho_0(ds)
\end{aligned}$$

Remark that by (1) $s \times \tau \to {}_\tau H_\Lambda(s) - {}_\tau H_\Lambda(Us) \in C(K^\infty)$. By convexity of $t \to \exp t$; applying Jenssen's inequality

$$\rho(\tilde{U}f) \geqslant \rho(f)\exp \frac{\rho(f(H - H \cdot U))}{\rho(f)} \quad \blacksquare$$

This theorem yields a set of inequalities for correlation functions. Before relating them to the variational principle of statistical mechanics we want to examine how good they are. In fact, we will prove that they form a complete set by proving that any state satisfying these inequalities necessarily must satisfy the DLR equations.

In order to prove this statement we need that there are enough elements in the set Q. Therefore we assume the following condition on the lattice site phase space K.

Condition. We suppose that there exists a compact group G of homeomorphisms of K acting transitively on K, i.e., for all $x, y \in K$ there exists a $U \in G$ such that $Ux = y$.

As a consequence of this assumption there exists a unique G-invariant probability measure ρ_0 on K which defines the *a priori* measure. Remark that in all the current models this condition is satisfied.

Now we denote by Q_0 the group of transformations on K^∞ generated by the products $\prod_{j \in \mathbb{Z}^\nu} U_j$ where only a finite number of $U_j \in G_j$ are different from the neutral element and where G_j is a copy of G at the jth site.

Theorem 2.3. If ρ is a state of $C(K^\infty)$ such that for all $U \in Q_0$ and all positive $f \in C(K^\infty)$

$$\rho(f)\ln \frac{\rho(f)}{\rho(\tilde{U}f)} \leqslant \rho\big(f(\tilde{U}^{-1}H - H)\big)$$

then ρ satisfies the DLR equations.

Proof. From the inequalities, if K^{Λ^c} is left pointwise invariant under $U \in Q_0$, then for all $f \in C(K^\infty), f \geqslant 0$

$$\rho(f) \leqslant M(\Lambda)\rho(\tilde{U}f)$$

and

$$\rho(\tilde{U}f) \leqslant M(\Lambda)\rho(f)$$

where $M(\Lambda) = \exp 2|\Lambda| \cdot |||\phi|||$.

Hence the measure ρ is absolutely continuous with respect to the measure $\rho \cdot \tilde{U}$. Therefore, letting Q_0^Λ be the subgroup of Q_0 leaving the space K^{Λ^c} pointwise invariant and $d\nu_\Lambda$ the Haar measure of Q_0^Λ, then

$$\rho(f) \leqslant M(\Lambda)\int_{Q_0^\Lambda} d\nu_\Lambda(U)\rho(\tilde{U}f)$$

for all $f \in C(K^\Lambda), f \geqslant 0$. As ρ_0 is the unique Q_0^Λ-invariant measure, one has

$$\rho(f) \leqslant M(\Lambda)\rho_0(f) \quad \text{and also} \quad \rho_0(f) \leqslant M(\Lambda)\rho(f)$$

Hence the restrictions of ρ and ρ_0 to $C(K^\Lambda)$ are mutually absolutely continuous. For all $f \in C(K^\infty)$

$$\rho(f) = \int_{s\in K^\Lambda}\int_{\tau \in K^{\Lambda^c}} f(s \times \tau)\rho(s, d\tau)\rho_0(ds)$$

and for all $f \geqslant 0$ the inequality can be written as

$$\frac{\rho(f)}{\rho(\tilde{U}f)} \leqslant \exp \frac{\int_{K^\Lambda}\int_{K^{\Lambda^c}} f(s\times\tau)\big[{}_\tau H_\Lambda(Us) - {}_\tau H_\Lambda(s)\big]\rho(s, d\tau)\rho_0(ds)}{\int_{K^\Lambda}\int_{K^{\Lambda^c}} f(s \times \tau)\rho(s, d\tau)\rho_0(ds)}$$

Let s_0 and $s_1$be arbitrary elements of K^Λ and $U \in Q_0^\Lambda$ such that $Us_0 = s_1$. Using the continuity of $s \times \tau \to {}_\tau H_\Lambda(Us) - {}_\tau H_\Lambda(s)$ for all finite Λ, by taking for f a δ-convergent sequence with carrier $s_0 \times \tau_0$ one gets $\rho_0(ds)$ almost everywhere:

$$\rho(s_0, d\tau_0) \leqslant \rho(s_1, d\tau_0)\exp\big[{}_{\tau_0}H_\Lambda(s_1) - {}_{\tau_0}H_\Lambda(s_0)\big]$$

By interchanging s_0 and s_1 one gets the reverse inequality, hence the equality. This proves the theorem. ■

It is clear from this theorem that the inequality can be used as a complete characterization of an equilibrium state, also in the case of non-translation invariant states.

3. VARIATIONAL PRINCIPLE

We now give a physical motivation of the correlation inequalities by deriving them, for translation-invariant states, from the variational principle of statistical mechanics.

Let E be the set of translation-invariant states locally absolutely continuous with respect to ρ_0, i.e., for each finite $\Lambda \subset \mathbb{Z}^\nu$, $\rho(f) = \rho_0(\rho_\Lambda f)$, $f \in C(K^\Lambda)$, where $\rho_\Lambda \geqslant 0$ and $\rho_0(\rho_\Lambda) = 1$. The entropy density is given by

$$s(\rho) = \lim_{|\Lambda| \to \infty} -\frac{1}{|\Lambda|}\rho(\ln \rho_\Lambda)$$

where $|\Lambda|$ is the number of lattice points in Λ. The map $\rho \to s(\rho)$ is an affine upper semicontinuous map for the w^* topology.[13]

The energy density is given by

$$e(\rho) = \lim_{|\Lambda|\to\infty} \frac{\rho(H_\Lambda)}{|\Lambda|} = \rho(A_\phi)$$

where

$$A_\phi = \sum_{0 \in X} \frac{\phi(X)}{|X|} \in C(K^\infty)$$

Now any state $\rho \in \mathsf{E}$ is said to satisfy the variational principle if it minimizes the free energy: $f(\rho) = e(\rho) - s(\rho)$.

The idea is the following: if ρ satisfies the variational principle we perturb ρ in a translation-invariant way by applying a Markov semigroup. Then we look for an upper bound for the first-order variation of the free energy. For quantum mechanical systems this idea was developed before in Refs. 10 and 11, for continuous classical systems in Ref. 14.

Therefore we have to specify first of all the semigroups.

We consider the following set of generators (a subset of those of Ref. 15): for any $\Lambda \subset \mathbb{Z}^\nu$ let $0 \leqslant f \in C(K^\Lambda)$ and $U \in Q_0^\Lambda$, define $\mathfrak{L}$ for all local observables g:

$$(\mathfrak{L}g)(x) = \sum_{i \in Z^\nu} \tau_i f(x)(\tilde{U}_i - 1)g(x)$$

where $\tilde{U}_i = \tau_i \tilde{U} \tau_i^{-1}$.

For notational convenience we omit the f and U dependence of the generator $\mathfrak{L}$. The closure of $\mathfrak{L}$ generates a strongly continuous one-parameter unity preserving contraction semigroup of positive maps $\{e^{\lambda \mathfrak{L}}\}_{\lambda \in \mathbb{R}^+}$.[15]

Lemma 3.1. Under the assumptions of above, we have the following: (i) The energy per lattice site A_ϕ belongs to the domain of the generator $\mathfrak{L}$. (ii) For any $\rho \in \mathsf{E}$

$$\lim_{\lambda \to 0^+} \rho\left(\frac{e^{\lambda \mathfrak{L}} - 1}{\lambda} A_\phi \right) = \rho\big(f(\tilde{U} \cdot H - H)\big)$$

Proof. An easy estimate yields the convergence of the formal sequence

$$\sum_{i \in Z^\nu} \tau_i f((\tilde{U}_i - 1)A_\phi)$$

In fact, it is bounded by $2|\Lambda|\ \|f\|\ |||\phi|||$. Furthermore, let $\{\Lambda_n\}_n$ be an increasing, absorbing set of cubes, and

$$(A_\phi)_n = \sum_{0 \in X \subseteq \Lambda_n} \frac{\phi(X)}{|X|}$$

then

$$\|(A_\phi)_n - A_\phi\| \leqslant \sum_{0 \in X \subset \Lambda_n} \|\phi(X)\| \to 0 \qquad \text{if} \quad n \to \infty$$

and

$$\|\mathfrak{L}((A_\phi)_n) - \mathfrak{L}((A_\phi)_{n'})\| \leqslant 2\|f\|\,|\Lambda| \sum_{0 \in X \cap \Lambda_{n'} \setminus \Lambda_n \neq \varnothing} \|\phi(X)\|$$

This proves (i).

Using translation invariance, the definition of A_ϕ and the estimates of above,

$$\rho(\mathfrak{L}(A_\phi)) = \sum_{i\in\mathbb{Z}^\nu} \rho\left[f(\tilde{U}-1) \sum_{\substack{i\in X\\ X\cap\Lambda\neq\phi}} \frac{\phi(X)}{|X|} \right]$$

$$= \rho\Big(f \sum_{X\cap\Lambda\neq\phi} (\tilde{U}-1)\phi(X) \Big)$$

$$= \rho(f(\tilde{U}H - H)) \quad \blacksquare$$

Lemma 3.2. Under the assumptions above

$$\varliminf_{\lambda\to 0^+} \frac{s(\rho\cdot T_\lambda) - s(\rho)}{\lambda} \geqslant \rho(f)\ln \frac{\rho(f)}{\rho(\tilde{U}^{-1}f)}$$

Proof. Consider $\{\Lambda_n\}_n$ an increasing absorbing sequence of cubes. Denote by

$$\mathfrak{L}_n = \sum_{\substack{i\in\mathbb{Z}^\nu\\ i+\Lambda\subset\Lambda_n}} \tau_i f(\tilde{U}_i - 1)$$

where $f\in C(K^\Lambda)$ and $U\in Q_0^\Lambda$ for some finite $\Lambda\subset\mathbb{Z}^\nu$. Then $C(K^{\Lambda_n})$ is left globally invariant under $\exp\lambda\mathfrak{L}_n$. For any $\rho\in\mathsf{E}$, let ρ_n be the local weight functions of ρ restricted to Λ_n with respect to ρ_0. Define the map ϕ from $(\mathbb{R}^+,\mathbb{R}^+)$ to $\overline{\mathbb{R}}$:

$$\begin{aligned} \phi(u,v) &= u \ln\frac{u}{v} && \text{if } u>0,\ v>0\\ &= 0 && \text{if } u=0,\ v\geqslant 0\\ &= \infty && \text{if } u>0,\ v=0 \end{aligned}$$

We first consider the case that the state ρ is separating, i.e., for all $f\in C(K^\infty)$, $f\neq 0$ we have $\rho(f^2)>0$.

By the joint convexity of the function: $(u,v)\to\phi(u,v)$ one gets

$$-\rho_n(\mathfrak{L}_n(\ln\rho_n)) \geqslant \rho_n\Big(\sum_{i+\Lambda\subset\Lambda_n}\tau_i f\Big)\ln\frac{\rho_n\Big(\sum_{i+\Lambda\subset\Lambda_n}\tau_i f\Big)}{\rho_n\Big(\sum_{i+\Lambda\subset\Lambda_n}\tau_i\tilde{U}^{-1}f\Big)} \qquad (*)$$

Let $(\rho_n\exp\lambda\mathfrak{L}_n)_p$ be the periodic extension to $C(K^\infty)$ of the state $\rho_n\exp\lambda\mathfrak{L}_n$ of $C(K^{\Lambda_n})$, and denote by $\widetilde{(\rho_n\exp\lambda\mathfrak{L}_n)_p}$ its translation invariant mean. Clearly,

$$w^* - \lim_n \widetilde{(\rho_n\cdot\exp\lambda\mathfrak{L}_n)_p} = \rho\cdot\exp\lambda\mathfrak{L}$$

Using the affinity of the entropy density on the set of periodic states and its upper semicontinuity

$$\varlimsup_{n} \frac{S(\rho_n \cdot \exp \lambda \mathfrak{L}_n)}{|\Lambda_n|} = \varlimsup_{n} s\big((\rho_n \cdot \exp \lambda \mathfrak{L}_n)_p\big)$$

$$= \varlimsup_{n} s\Big(\widetilde{(\rho_n \cdot \exp \lambda \mathfrak{L}_n)_p}\Big) \leqslant s(\rho \cdot \exp \lambda \mathfrak{L})$$

Therefore

$$\varlimsup_{\lambda \to 0^+} \varlimsup_{n} \frac{S(\rho_n \cdot \exp \lambda \mathfrak{L}_n) - S(\rho_n)}{\lambda |\Lambda_n|} \leqslant \varlimsup_{\lambda \to 0^+} \frac{s(\rho \cdot \exp \lambda \mathfrak{L}) - s(\rho)}{\lambda}$$

Furthermore, using the inequality (*) derived above, with the notation of $\mathfrak{L}^*$, $\rho_0(\mathfrak{L}^*(f)g) = \rho_0(f\mathfrak{L}(g))$; $f, g \in C(K^\infty)$ then

$$\varlimsup_{\lambda \to 0^+} \varlimsup_{n} \frac{S(\rho_n \cdot \exp \lambda \mathfrak{L}_n) - S(\rho_n)}{\lambda |\Lambda_n|}$$

$$= \varlimsup_{\lambda \to 0^+} \varlimsup_{n} -\frac{1}{\lambda} \int_0^\lambda \frac{ds}{|\Lambda_n|} \int_{K^{\Lambda_n}} \rho_0(dx) \frac{d}{ds} \left\{ \exp s\mathfrak{L}^*{}_n(\rho_n) \ln\left[\exp s\mathfrak{L}^*{}_n(\rho_n)\right]\right\}$$

$$= \varlimsup_{\lambda \to 0^+} \varlimsup_{n} -\frac{1}{\lambda} \int_0^\lambda \frac{ds}{|\Lambda_n|} \int_{K^{\Lambda_n}} \rho_0(dx) \exp s\mathfrak{L}^*{}_n(\rho_n) \mathfrak{L}_n\left\{\ln\left[\exp s\mathfrak{L}^*{}_n(\rho_n)\right]\right\}$$

$$\geqslant \varlimsup_{\lambda \to 0^+} \varlimsup_{n} \frac{1}{\lambda} \int_0^\lambda ds \frac{\rho_n\left[\exp(s\mathfrak{L}_n)\left(\sum_{i+\Lambda \subset \Lambda_n} \tau_i f\right)\right]}{|\Lambda_n|}$$

$$\times \ln \frac{\rho_n\left[\exp(s\mathfrak{L}_n)\left(\sum_{i+\Lambda \subset \Lambda_n} \tau_i f\right)\right]}{\rho_n\left[\exp s\mathfrak{L}_n\left(\sum_{i+\Lambda \subset \Lambda_n} \tau_i \tilde{U}^{-1} f\right)\right]} \qquad (**)$$

Next we remark on the continuity of

$$s \to \psi(s,n) = \frac{1}{|\Lambda_n|} \rho_n\left[\exp(s\mathfrak{L}_n)\left(\sum_{i+\Lambda \subset \Lambda_n} \tau_i f\right)\right]$$

uniformly in n. Indeed

$$|\psi(s_1,n) - \psi(s_2,n)| = \left| \int_{s_2}^{s_1} ds \frac{\rho_n\left[\exp(s\mathfrak{L}_n)\mathfrak{L}_n\left(\sum_{i+\Lambda \subset \Lambda_n} \tau_i f\right)\right]}{|\Lambda_n|} \right|$$

$$\leqslant |s_1 - s_2| 2 \|f\|^2 |\Lambda|^2$$

As ρ_n is separating and as the function ϕ is continuous on $([0, \infty), (0, \infty))$, the integrand of (**) is continuous in the variable s, uniformly in n. By interchanging the limits, using the mean value theorem, one gets

$$\underline{\lim}_{\lambda \to 0^+} \frac{s(\rho \cdot e^{\lambda \mathcal{L}}) - s(\rho)}{\lambda} \geqslant \rho(f) \ln \frac{\rho(f)}{\rho(\tilde{U}^{-1} f)}$$

Finally one has to remove the requirements of ρ_n being separating. As ρ_0 is invariant under Q_0^Λ one readily checks that it is separating. Now if ρ is not separating, consider then the state $\rho_\tau = (1 - \tau)\rho + \tau \rho_0$, $\tau \in (0, 1)$. Clearly, ρ_τ is separating and from above we have

$$\underline{\lim}_{\lambda \to 0^+} \frac{s(\rho_\tau \cdot e^{\lambda \mathcal{L}}) - s(\rho_\tau)}{\lambda} \geqslant \rho_\tau(f) \ln \frac{\rho_\tau(f)}{\rho_\tau(\tilde{U}^{-1} f)}$$

From the affinity of the entropy density and the fact that $s(\rho_0 \cdot e^{\Lambda \mathcal{L}}) \leqslant s(\rho_0) = 0$ we get the desired result. ■

Now we are in a position to prove the following:

Theorem 3.3. If ρ is a state in E satisfying the variational principle, then for all positive elements f of $C(K^\infty)$ and all $U \in Q_0^\Lambda$, Λ an arbitrary finite volume, we have

$$\rho(f) \ln \frac{\rho(f)}{\rho(\tilde{U}^{-1} f)} \leqslant \rho(f(\tilde{U}H - H))$$

Proof. As ρ satisfies the variational principle;

$$\underline{\lim}_{\lambda \to 0^+} \rho\left(\frac{(e^{\lambda \mathcal{L}} - 1)}{\lambda} A_\phi \right) \geqslant \underline{\lim}_{\lambda \to 0^+} \frac{s(\rho \cdot e^{\lambda \mathcal{L}}) - s(\rho)}{\lambda}$$

The result follows now from Lemmas 3.1 and 3.2. ■

4. ILLUSTRATION

As an illustration of the kind of problems tractable by the inequality we consider the case of spin-1/2 systems (i.e., $K = \{1, -1\}$).

Simon and Sokal[7] derive among others some results on one-dimensional long-range models. Their method consists in proving that appropriate, absolutely nontrivial, sets of configurations are excluded on the basis of thermodynamic stability. From Section 3, one might expect our inequality to do a similar job. In fact, it turns out that we do not have to deal explicitly with configurations but only to make a suitable, yet natural choice of a positive function. Moreover, as the inequality is immediately given for the infinite system, we avoid troublesome estimations induced by boundary effects.

On the other hand, Fröhlich and Pfister (3) prove, using the concept of relative entropy, the absence of spontaneous breaking of the internal and spatial symmetry in classical two-dimensional systems. Below we illustrate our method for this kind of problems by proving translation invariance for one-dimensional long-range spin-1/2 systems.

These illustrations deal essentially with absence of symmetry breaking. It would also be interesting to prove the existence of these phenomena (see, e.g., Ref. 16) along these lines.

Let

$$H_\Lambda = -\beta \sum_{i<j\in\Lambda} J_{ij}\sigma_i\sigma_j, \qquad \Lambda \subset \mathbb{Z}$$

where σ_i stands for the function σ on the ith site, taking the values ± 1 on $K = \{-1, 1\}$ and where $J_{ij} \in \mathbb{R}$. We impose the usual conditions:

(i) $$J_{ij} = J(|i-j|), \qquad J(0) = 0$$

(ii) $$\sum_{j=1}^{\infty} |J(j)| < \infty$$

The expectation values in an equilibrium state ρ will be denoted by the brackets: $\rho(A) = \langle A \rangle$, $A \in C(K^\infty)$.

Proposition 4.1[7] If the interaction satisfies

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} k|J(k)|}{\ln N} = 0$$

and if the translation-invariant state satisfies

$$\frac{1}{4N^2} \sum_{k,l=-N}^{N} (\langle \sigma_k \sigma_l \rangle - \langle \sigma_0 \rangle^2) \leqslant \frac{C}{N^\alpha}$$

for some C and $\alpha > 0$, then $\langle \sigma_0 \rangle = 0$.

Proof. Substitute in the inequality 2.2 for f the function

$$f_N = \frac{1}{4N^2} \left[\sum_{k=-N}^{N} (\sigma_k + \langle \sigma_0 \rangle) \right]^2$$

and for U the operation U_N flipping all spins of the interval $[-N, N]$; then

$$\langle \tilde{U}_N f_N \rangle = \frac{1}{4N^2} \sum_{k,l=-N}^{N} \left(\langle \sigma_k \sigma_l \rangle - \langle \sigma_0 \rangle^2 \right) \leqslant \frac{C}{N^\alpha}$$

also

$$\langle f_N \rangle = 4\langle \sigma_0 \rangle^2 + \langle \tilde{U}_N f_N \rangle \geqslant 4\langle \sigma_0 \rangle^2$$

and

$$|\langle f_N (H \circ U_N - H) \rangle| \leqslant \langle f_N \rangle \| H \circ U_N - H \|$$

Now

$$\| H \circ U_N - H \| \leqslant 4\beta \left[\sum_{k=1}^{N} |kJ(k)| + N \sum_{k=N+1}^{\infty} |J(k)| \right]$$

Using the following estimate

$$N \sum_{k=N+1}^{M} |J(k)| \leqslant \frac{N}{M} \sum_{k=1}^{M} k|J(k)| + N \sum_{l=N+1}^{M-1} \frac{1}{l^2} \sum_{k=1}^{l} k|J(k)|$$

one checks immediately that

$$\lim_{N\to\infty} \lim_{M\to\infty} \frac{N}{\ln N} \sum_{k=N+1}^{M} |J(k)| = 0$$

follows from the condition on the interaction. Gathering all this information in the inequality one gets

$$4\langle \sigma_0 \rangle^2 \ln \frac{4\langle \sigma_0 \rangle^2 N^\alpha}{C} \leqslant 16\beta \langle \sigma_0 \rangle^2 \left[\sum_{k=1}^{N} k|J(k)| + N \sum_{k=N+1}^{\infty} |J(k)| \right]$$

Dividing both sides of this inequality by $\ln N$ and taking the limit of N tending to infinity, one gets a result contradicting $\langle \sigma_0 \rangle \neq 0$. ■

Remark that the same proof can be used to derive that all odd-point functions vanish under the same condition on the interaction and an analogous cluster property.

It follows also from the proof that the specific decay rate of the clustering is irrelevant if the interaction satisfies the more stringent condition $\sum_{k=1}^{\infty} k|J(k)| < \infty$.

Now we prove spatial invariance of the equilibrium state. The proof uses mainly the inequality to prove absolute continuity of the state with respect to its translate. Concerning the idea of working toward proving relative absolute continuity of states one should refer to Refs. 2, 3, 17, and 18.

Proposition 4.2. If the interaction satisfies

$$\operatorname*{Sup}_N \left[\sum_{k=1}^{N} k|J(k) - J(k-1)| + N \sum_{k=N+1}^{\infty} |J(k) - J(k-1)| \right] < \infty$$

then any equilibrium state is spatially invariant.

Proof. Take in the inequality 2.2 for $\tilde{U}$ the map $\tilde{U}_N$ of the interval $[-N, N]$ into itself given by

$$\begin{aligned} \tilde{U}_N \sigma_i &= \sigma_{i+1} \qquad \text{if} \quad i = -N, \ldots, N-1 \\ \tilde{U}_N \sigma_N &= \sigma_{-N} \\ \tilde{U}_N \sigma_j &= \sigma_j \qquad \text{if} \quad |j| > N \end{aligned}$$

Then for any positive function f of $C(K^{\infty})$ one has again

$$|\langle f(\tilde{U}_N \circ H - H)\rangle| \leqslant \langle f \rangle \| \tilde{U}_N \circ H - H \|$$

Now by an elementary but tedious counting one gets

$$\| \tilde{U}_N \circ H - H \| \leqslant \ln C$$

where

$$\ln C = \beta \left[12 \sum_{k=1}^{\infty} |J(k)| + 2 \sum_{k=1}^{\infty} k|J(k) - J(k-1)| \right]$$

is independent of N.

After substitution of this bound in the inequality

$$\ln \frac{\langle f \rangle}{\langle \tilde{U}_N^{-1} f \rangle} \leqslant \ln C$$

Remark that

$$\lim_{N \to \infty} \langle \tilde{U}_N f \rangle = \langle \tau_1 f \rangle$$

where τ_1 is the translation over one lattice site. Hence for all positive $f \in C(K^{\infty})$,

$$C^{-1} \langle \tau_1 f \rangle \leqslant \langle f \rangle \leqslant \langle \tau_1 f \rangle C$$

It is by now a standard argument (see, e.g., Refs. 17 and 18) to conclude that two extremal DLR states which are mutually absolutely continuous coincide. ■

Remark that the condition on the potential is automatically satisfied in the case of ferromagnetic or antiferromagnetic systems if the function $k \to J(k)$ is monotone for k large enough.

Finally, we close with an illustration about the exponential decay of correlations for the ν-dimensional Ising model in a cubic lattice.

Denote

$$\Lambda_n = \{z \in \mathbb{Z}^\nu | 1 \leqslant z_1 \leqslant n; z_i = 0 \text{ for } i \neq 1\}$$

and

$$m = -\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \frac{1}{n^2} \left\langle \left[\sum_{k\in\Lambda_n} (\sigma_k - \langle\sigma_0\rangle) \right]^2 \right\rangle$$

for any translation-invariant equilibrium state.

Proposition 4.3 (Schor (19)). If $\langle\sigma_0\rangle \neq 0$ then $m \leqslant 4(\nu - 1)\beta J$.

Proof. We use again 2.2 with again the positive function

$$f_{\Lambda_n} = \frac{1}{n^2} \left[\sum_{k\in\Lambda_n} (\sigma_k + \langle\sigma_0\rangle) \right]^2$$

and again U_n the flip of all spins in Λ_n. Substitute the easy estimate

$$\| \tilde{U}_n \circ H - H \| \leqslant 4(\nu - 1)\, \beta J n + 4\beta J$$

in the inequality. Divide both sides by n and take the limit n tending to infinity. ■

For small temperatures, this result is optimal. Indeed, in that case the upper bound obtained here coincides with the lower bound as obtained from the Peierls argument.

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